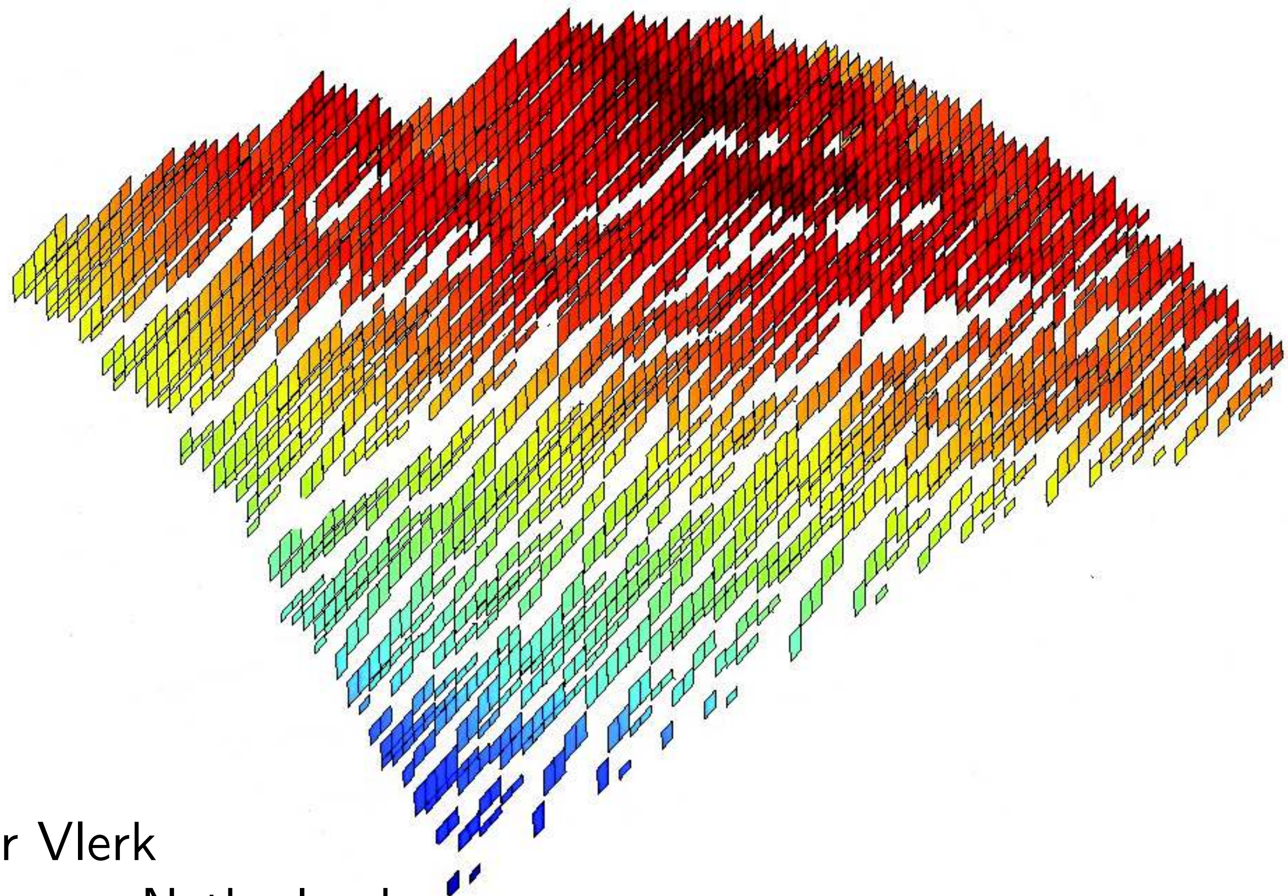
Stochastic Mixed-Integer Programming

Tutorial SPXII, Halifax, August 15, 2010



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Outline

Stochastic Mixed-Integer Programming

- introduction & motivation
- Simple Integer Recourse: definition, properties (later: more general)
- convex approximations SIR $\longrightarrow$ solution methods
- same approach: Complete IR

Decomposition:

- Benders and beyond
- Dual decomposition (risk models)

Mention:

- large-scale deterministic MIP
- Decomposition based B&B
- Branch-and-Fix Coordination
- . . .

Stochastic Mixed-Integer Problems (two-stage)

Generalization of continuous recourse model:

$$\min_{x \in X} \{cx + Q(x) : Ax = b\}$$

where

$$Q(x) = \mathbb{E}_\omega \left[v\left(h(\omega) - T(\omega)x\right) \right], \qquad v(s) = \min_{y \in Y} \{qy : Wy = s\}$$

with $X = \mathbb{Z}_+^{\bar{n}} \times \mathbb{R}_+^{n-\bar{n}}$, $\quad Y = \mathbb{Z}_+^{\bar{p}} \times \mathbb{R}_+^{p-\bar{p}}$ (canonical form)

Terminology: Q expected value function (EVF), v second-stage value function, W recourse matrix

Classification SMIP problems [Sen '04]

- $B \equiv$ {set of stages with *binary* decisions}
- $C \equiv$ {set of stages with *continuous* decisions}
- $D \equiv$ {set of stages with *discrete* decisions}

SMIP: $B \cup D \neq \emptyset$, focus on $2 \in B \cup D$

Why include integer variables?

- natural integrality of decision variables
  e.g. *The Allocation of Aircraft to Routes* [Ferguson & Dantzig '56]
- yes/no, on/off decisions $\longrightarrow \{0,1\}$ variables
- artificial indicator variables for conditional linear constraints (LP formulation of CO problems)

$$0 \leq x \leq Mz, \quad x \in \mathbb{R}, \quad z \in \{0,1\}$$

- satisfy k out of n constraints, e.g. discrete Chance Constraints

$$\Pr\{Tx \geq \omega\} \geq \alpha \in (0,1) \text{ with } \Pr\{\omega = \omega^s\} = p^s,\ s = 1, \ldots, S$$

Why not?

- continuous SLP is already difficult enough
- complexity: 2nd-stage problems NP-hard

Theoretical argument: continuous SLP is #P complete [Dyer & Stougie '06]
$\longrightarrow$ SMIP not harder (. . .)

Some SMIP applications

sequencing	Jørnsten ('92)
scheduling	Birge, Dempster ('96), Tayur, Thomas, Natraj ('95)
routing	Laporte, Louveaux, Mercure ('92)
location	Laporte, Louveaux, Van Hamme ('94)
unit commitment	Takriti, Birge, Long ('96), Carøe, Ruszczynski, Schultz ('97)
	Philpott, Römisch, Schultz, Escudero ea, . . .
dispersed generation	Klein Haneveld, VdV ('00), Gollmer et al. ('07)
pollution control	Ruszczynski, Ermoliev, Norkin ('95)
ALM	Dert ('95), Drijver, KH, VdV ('03), Streutker, KH, VdV ('10)
stochastic GAP	Albareda, VdV, Fernandez ('03)
network expansion	Norkin ea ('95)
interdiction	Morton, Pan ('04), Poss ('09)
design	Crainic, Lium, Wallace ('04)
natural gas value chain opt.	Tomasgard, Fodstad ('04)
lot sizing	Guan, Ahmed, Nemhauser ('06)
capacity expansion	Ahmed, King, Parija, Gyana ('03)
paratransit service	Higle ('04), Cremers, KH, VdV ('10)
many more	see e.g. SPXII program

See S(I)P Bibliography and SP E-Print Series via COSP website `http://stoprog.org`

Borrow from solution approaches for deterministic MIP:

- LP + rounding: no good
- Branch & Bound with LP relaxation
- Benders' decomposition
- Polyhedral theory: valid inequalities
- Lagrangian relaxation

Combine with SLP algorithms $\longrightarrow$ algorithms for SMIP? 2nd part

Borrow from solution approaches for deterministic MIP:

- LP + rounding: NO GOOD
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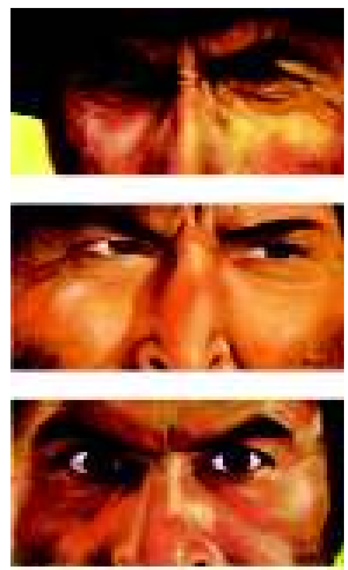
Combine with SLP algorithms $\longrightarrow$ algorithms for SMIP? 2nd part

First: SMIP is a battle between

- randomness: *Good*
- integrality: *Bad*

Usually, result is *Ugly*: non-convex, . . .

Sometimes result is *Beautiful*: convex!



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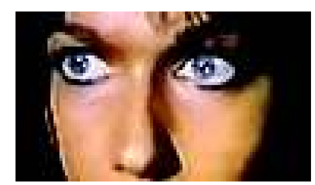


!

Simple Integer Recourse

By definition, the second-stage problem is

$$v(s) := \min_{y^+, y^-} \{q^+ y^+ + q^- y^- : \quad \begin{aligned} & y^+ \geq s \\ & y^- \geq -s \\ & y^+ \in \mathbb{Z}^m_+, \; y^- \in \mathbb{Z}^m_+ \; \} \end{aligned}$$

with $q^+ \geq 0$, $q^- \geq 0$ $\quad \longrightarrow v > -\infty$: sufficiently expensive

Special case: $T(\omega) = T$ fixed, $h(\omega) = \omega$, so that $Q(x) = \mathbb{E}_\omega \left[v(\omega - Tx) \right]$

SR structure $\longrightarrow v$ and Q separable in tender variables $s = Tx$

$$v(s) \quad = \quad \sum_{i=1}^{m} \left(q_i^+ \lceil s_i \rceil^+ + q_i^- \lfloor s_i \rfloor^- \right), \qquad s \in \mathbb{R}^m$$

where $\lceil t \rceil^+ := \max\{0, \lceil t \rceil\}$ and $\lfloor t \rfloor^- := \max\{0, -\lfloor t \rfloor\}$

Q also separable

$$Q(s) = \sum_{i=1}^{m} \mathbb{E}_{\omega_i}\left[q_i^+ \lceil \omega_i - s_i \rceil^+ + q_i^- \lfloor \omega_i - s_i \rfloor^-\right], \qquad s \in \mathbb{R}^m$$

Consider instead one-dimensional generic SIR functions

$$v(s) = q^+ \lceil s \rceil^+ + q^- \lfloor s \rfloor^-, \qquad s \in \mathbb{R}$$

$$Q(x) = q^+ \mathbb{E}_\omega\left[\lceil \omega - x \rceil^+\right] + q^- \mathbb{E}_\omega\left[\lfloor \omega - x \rfloor^-\right], \qquad x \in \mathbb{R}$$

with $\omega \in \Omega \subset \mathbb{R}$

Compare to continuous SR analogues

$$\bar{v}(s) = q^+ (s)^+ + q^- (s)^-$$

$$\bar{Q}(x) = q^+ \mathbb{E}_\omega\left[(\omega - x)^+\right] + q^- \mathbb{E}_\omega\left[(\omega - x)^-\right]$$

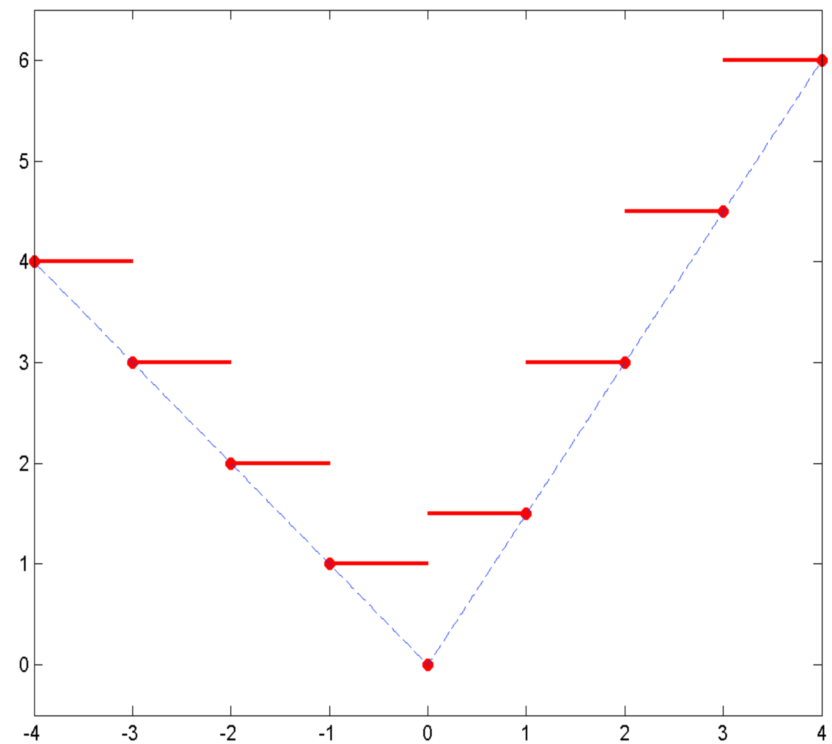
Same, except for rounding

Properties of Q via properties of

$$v(\omega - x) = q^+\lceil \omega - x\rceil^+ + q^-\lfloor \omega - x\rfloor^-, \quad x \in \mathbb{R}$$

with ω fixed

- discontinuous at $x = \omega + k$, $k \in \mathbb{Z}$
  jump q^+ at $x = \omega + k$, $k \in \mathbb{Z}_+$
  jump q^- at $x = \omega - k$, $k \in \mathbb{Z}_+$
- right continuous at $x > \omega$
  left continuous at $x < \omega$
  neither at $x = \omega$
- piecewise constant
- lower semicontinuous: $v(s) \leq \lim_{u \longrightarrow s} v(u)$
- non-convex



v and $\bar{v}$ (dashed), $q^+ = 1$, $q^- = 1.5$

One-dimensional generic SIR function Q [Louveaux & VdV '93, VdV '95]

Useful formula: for $x \in \mathbb{R}$

$$Q(x) = q^+ \mathbb{E}_\omega \left[\lceil \omega - x \rceil^+ \right] + q^- \mathbb{E}_\omega \left[\lfloor \omega - x \rfloor^- \right]$$

$$= q^+ \sum_{k=0}^{\infty} \Pr\{\omega > x + k\} + q^- \sum_{k=0}^{\infty} \Pr\{\omega < x - k\}$$

Compare to continuous SR analogue $\bar{Q}$:

$$\bar{Q}(x) = q^+ \mathbb{E}_\omega \left[(\omega - x)^+ \right] + q^- \mathbb{E}_\omega \left[(\omega - x)^- \right]$$

$$= q^+ \int_x^{\infty} \Pr\{\omega > s\} ds + q^- \int_{-\infty}^{x} \Pr\{\omega < s\} ds$$

Properties of Q

Like continuous SR: finite iff $\mathbb{E}_\omega[|\omega|] < +\infty$ (assume from now on)

Continuity

Recall: for ω fixed

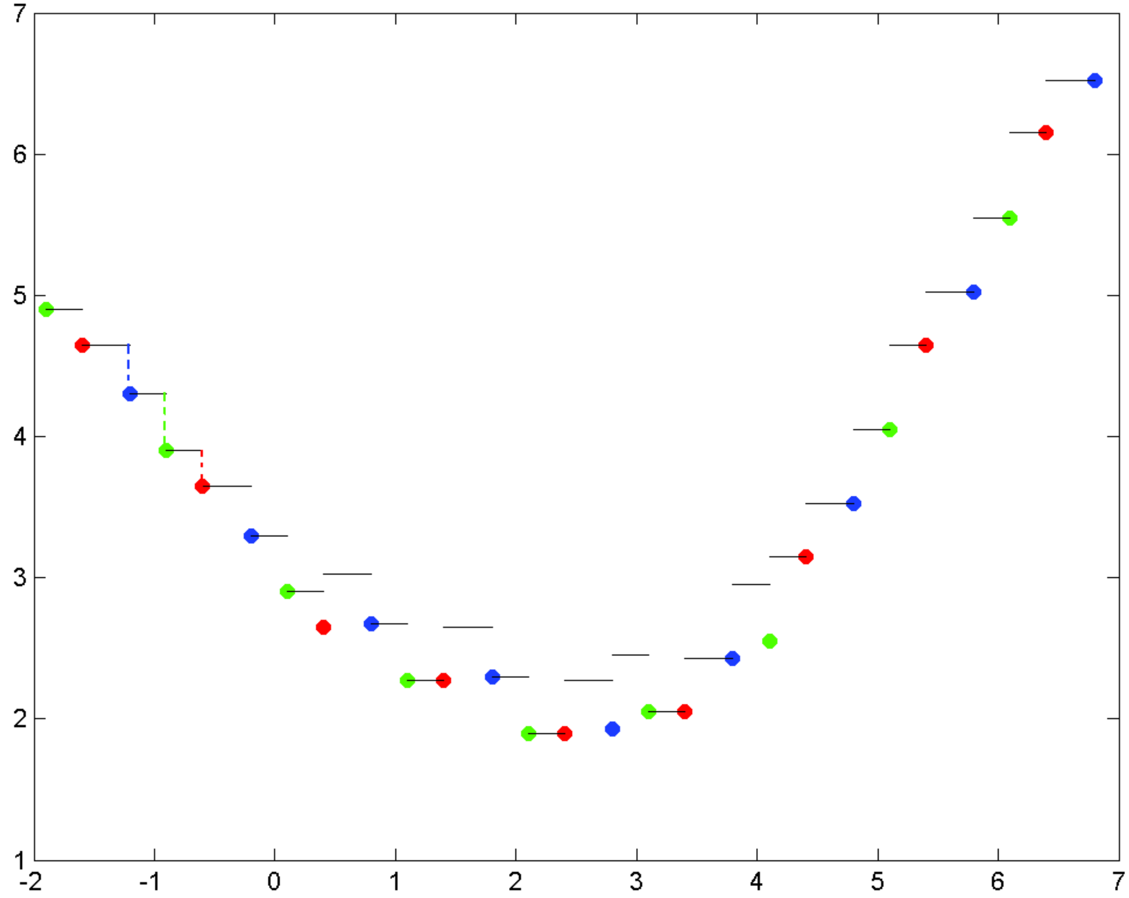
- v is lower semicontinuous
- $v(\omega - x)$ is discontinuous at $x = \omega + k, \; k \in \mathbb{Z}$
- constant in between

Q is lower semicontinuous for any distribution of ω

Q is discontinuous at x iff $\Pr\{\omega \in x + \mathbb{Z}\} > 0$

If ω is discretely distributed with support Ω then

- Q is discontinuous at $\Omega + \mathbb{Z}$
- constant in between

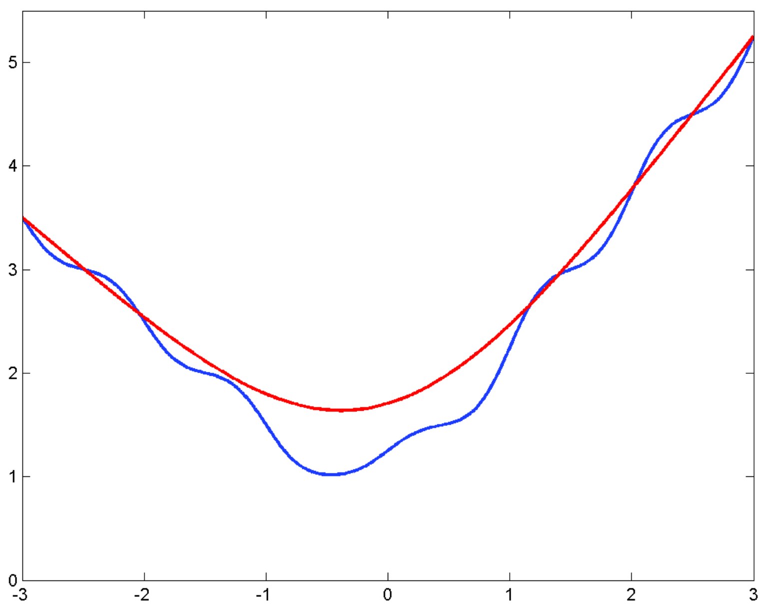


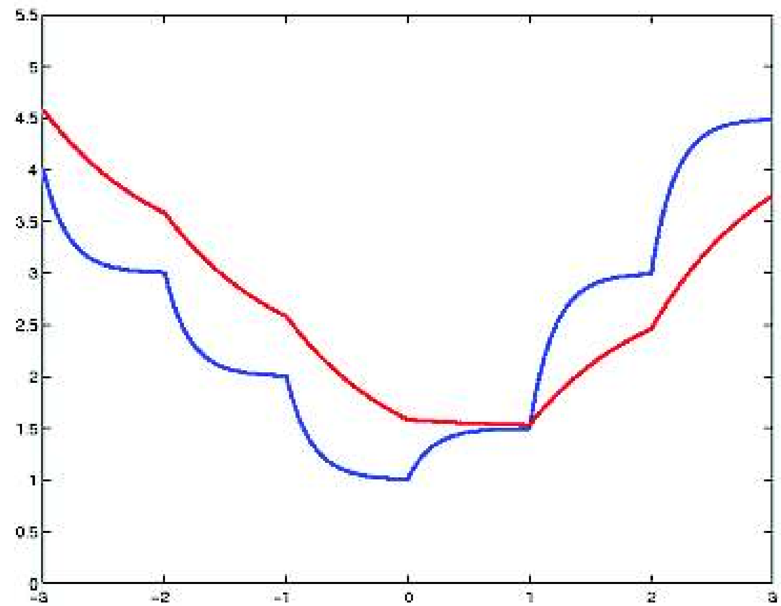
$$\Omega = \{0.4, 2.8, 4.1\} \quad p = (0.25, 0.35, 0.4)$$

$$q^+ = 1 \qquad q^- = 1.5$$

Recall: Q is discontinuous at x iff $\Pr\{\omega \in x + \mathbb{Z}\} > 0$

$\longrightarrow$ Q is continuous if and only if ω is continuously distributed





$\omega \sim N(0, 0.05)$ $\quad$ $q^{+} = 1,\ q^{-} = 1.5$ $\quad$ $\omega \sim \mathcal{E}(5)$

$\omega \sim N(0, 1)$ $\quad$ $\omega \sim \mathcal{E}(1)$

(Lipschitz continuity, differentiability)

Convexity

In general Q is non-convex: expectation of 'step function' $v(\omega - x)$

For all distributions of ω,
Q convex on $\alpha + \mathbb{Z}$, $\alpha \in [0,1)$

$\longrightarrow$ 'reasonable' convex approximations . . .

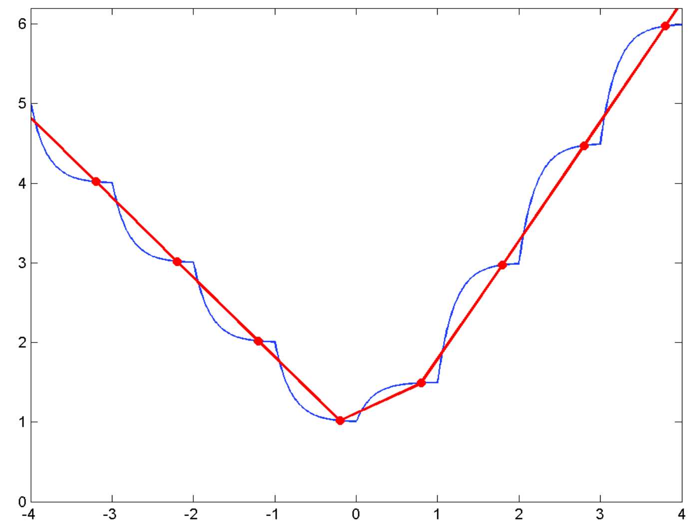
Q convex $\Rightarrow$ ω continuous, but $\not\Leftarrow$

Theorem [Klein Haneveld, Stougie, VdV '06]
SIR function Q is convex iff $\omega \sim$ pdf f with

$$f(s) = G(s+1) - G(s), \quad s \in \mathbb{R}$$

where G is an arbitrary cdf with finite mean

Example: for any $\alpha \in [0,1)$
G discrete on $\alpha + \mathbb{Z}$ $\longrightarrow$ f constant on $[\alpha + k, \alpha + k + 1)$, $k \in \mathbb{Z}$



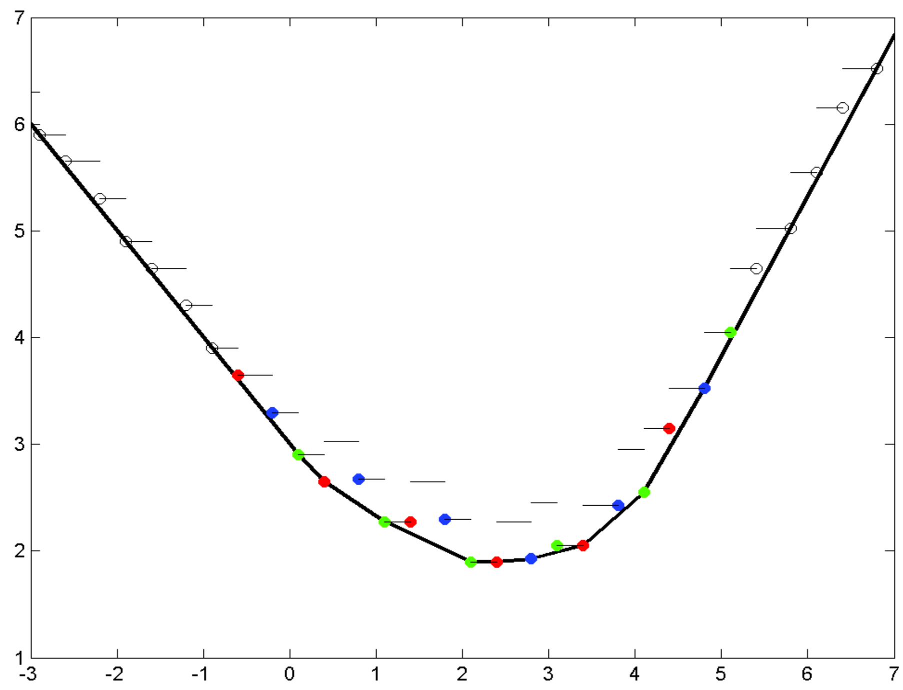
$\omega \sim \text{Exp}(5)$, $\alpha = 0.8$

Algorithms for SIR

Approach:

- construct convex approximation of $Q = \mathbb{E}_\omega[\ldots]$
- show: equivalent to Simple Continuous Recourse function $\bar{Q} = \mathbb{E}_\xi[\ldots]$

$\longrightarrow$ approximate solution using SCR algorithm



Discrete distribution RHS ω

[Klein Haneveld, Stougie, VdV '96]

- construct convex hull of Q
- strongly polynomial algorithm
  $\longrightarrow$ ξ in SCR
- convex hull EVF if T full row rank
- constraints $Ax = b$?

Continuous distribution RHS ω

To show: SIR 'solvable' as continuous SR
i.e. convex approximation of $Q = \mathbb{E}_{\omega}[\ldots]$ equals $\bar{Q} = \mathbb{E}_{\xi}[\ldots]$ for some rhs ξ

SR structure: v separable $\longrightarrow$ consider for $s \in \mathbb{R}$

$$v(s) \quad = \quad q^+ \lceil s \rceil^+ + q^- \lfloor s \rfloor^- \qquad (q^+, q^- \geq 0)$$

and SIR recourse function (tender $z_{\not i} = T_{\not i} x \in \mathbb{R}$)

$$Q(z) \quad := \quad \mathbb{E}_\omega [v(\omega - z)] = q^+ \mathbb{E}_\omega \big[\lceil \omega - z \rceil^+ \big] + q^- \mathbb{E}_\omega \big[\lfloor \omega - z \rfloor^- \big]$$

$\omega \in \Omega \subset \mathbb{R}$ with cdf F (pdf f), finite mean μ

Recall: Q convex iff $\omega \sim$ pdf f with

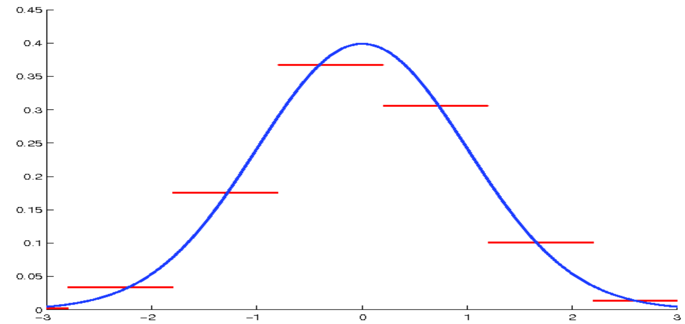
$$f(s) = G(s+1) - G(s), \quad s \in \mathbb{R}$$

where G is a cdf with finite mean

In particular: for arbitrary fixed $\alpha \in [0, 1)$

$$f \text{ constant on } [\alpha + k, \alpha + k + 1), \quad k \in \mathbb{Z}$$

($G \sim$ discrete on $\alpha + \mathbb{Z}$)



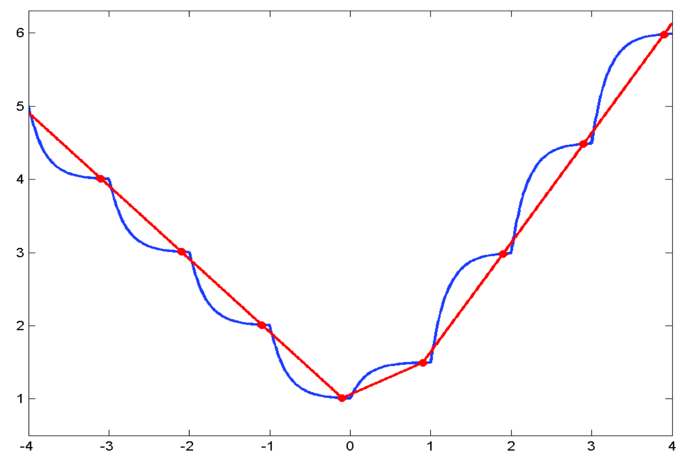
Approximate $\omega \sim F$ by $\omega_\alpha \sim f_\alpha$, $\alpha \in [0, 1)$

$$f_\alpha(t) := F(\lceil t \rceil_\alpha) - F(\lceil t \rceil_\alpha - 1), \quad t \in \mathbb{R}$$

with $\lceil t \rceil_\alpha := \lceil t - \alpha \rceil + \alpha$ round up wrt $\alpha + \mathbb{Z}$

Then the *α-approximation*

$$Q_\alpha(z) \quad := \quad q^+ \mathbb{E}_{\omega_\alpha}\left[\lceil \omega_\alpha - z \rceil^+\right] + q^- \mathbb{E}_{\omega_\alpha}\left[\lfloor \omega_\alpha - z \rfloor^-\right]$$



is a convex approximation of Q

Error bound: $\|Q_\alpha - Q\|_\infty \leq \max\{q^+, q^-\}\ \frac{TV(f)}{4}$

The convex function Q_α 'looks like' a continuous SR function . . .

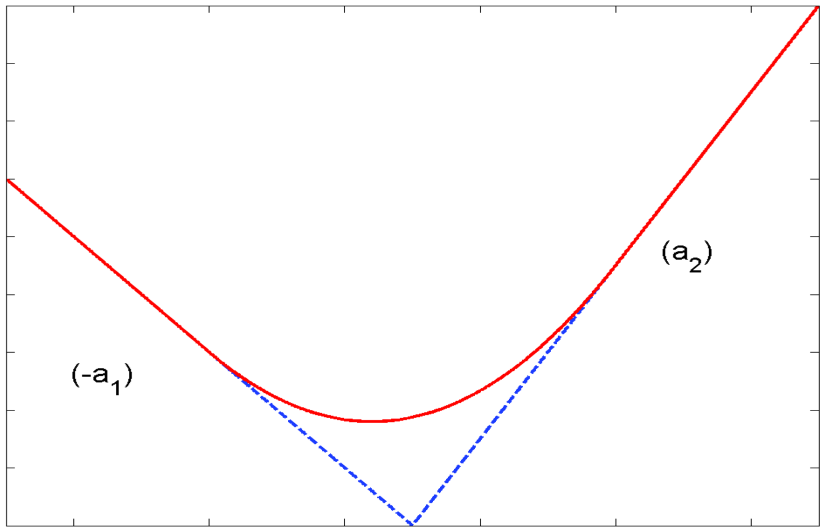
Theorem [KH, S, VdV '93]

Let $\varphi(s)$, $s \in \mathbb{R}$

- convex (non-linear)
- asymptotes with slopes

$-a_1$ as $s \to -\infty$

a_2 as $s \to \infty$



Then φ is a SCR function (+ known const.):

$$\varphi(s) = a_1 \mathbb{E}_\xi\left[(\xi - s)^+\right] + a_2 \mathbb{E}_\xi\left[(\xi - s)^-\right] + C$$

where ξ is a random variable with cdf Φ

$$\Phi(t) = \frac{\varphi'_+(t) + a_1}{a_1 + a_2}$$

Apply to Q_α: for $\alpha \in [0,1)$

$$Q_\alpha(z) = q^+ \mathbb{E}_{\xi_\alpha}\left[(\xi_\alpha - z)^+\right] + q^- \mathbb{E}_{\xi_\alpha}\left[(\xi_\alpha - z)^-\right] + \frac{q^+ q^-}{q^+ + q^-}$$

where ξ_α is discrete on $\alpha + \mathbb{Z}$ with cdf

$$\Phi_\alpha(t) = \frac{q^+ F(\lceil t \rceil_\alpha - 1) + q^- F(\lceil t \rceil_\alpha)}{q^+ + q^-}$$

and F is the cdf of ω

Conclusion: to (approximately) solve SIR

$$\left([q^+, q^-],\ W_{\text{SIR}},\ \mathbb{Z}_+^{2m},\ F\right)$$

solve continuous SR

$$\left([q^+, q^-],\ [I, -I],\ \mathbb{R}_+^{2m},\ \Phi_\alpha\right)$$

Observe: Φ_α discrete distribution

Modification of recourse data, also for other model classes [VdV '03]

Complete Integer Recourse

To show: ‘solvable’ as continuous CR

Assumptions:

- recourse complete & sufficiently expensive $\longrightarrow v$ finite
- $\mathbb{E}_\omega[||\omega||]$ finite $\longrightarrow Q$ finite

Integrality $\longrightarrow v$ and Q non-convex in x

Evaluation $Q(x)$: solve IP for each $\omega \in \Omega$

Special case: W is Totally Unimodular

$$\min_y \left\{ qy : Wy \geq s,\ y \in \mathbb{Z}^n_+ \right\} \qquad \text{(IP}(s))$$

e.g., network flow, shortest path, . . .

Then IP(s)=LP(s) (with integer solution) if right-hand side s is integer

However, in CIR right-hand side is $\omega - Tx \in \mathbb{R}^m$ . . .

Assume W is Totally Unimodular

For $s \in \mathbb{R}^m$ (!!)

$$\begin{aligned} v(s) &= \min_y \{qy : Wy \geq s,\ y \in \mathbb{Z}^n_+\} \\ &= \min_y \{qy : Wy \geq \lceil s \rceil,\ y \in \mathbb{R}^n_+\} \\ &= \max_\lambda \{\lambda \lceil s \rceil : \lambda W \leq q,\ \lambda \in \mathbb{R}^m_+\} \end{aligned}$$

Recourse complete & suff. expensive:

$\Lambda := \{\lambda \in \mathbb{R}^m_+ : \lambda W \leq q\}$ is bounded, non-empty $\longrightarrow$

$\Lambda = \text{conv}\{\lambda^1, \ldots, \lambda^K\}$, $\quad \lambda^k \geq 0$ extreme points

Hence

$$v(s) = \max_{k \in \{1,\ldots,K\}} \lambda^k \lceil s \rceil, \quad s \in \mathbb{R}^m$$

pointwise maximum of finitely many round-up functions $\lambda^k \lceil s \rceil$

Consider $\lambda\lceil s\rceil$, $\lambda \in \mathbb{R}^m_+$, and for $z \in \mathbb{R}^m$

$$\mathcal{R}(z) \quad := \quad \lambda\, \mathbb{E}_\omega[\lceil \omega - z\rceil] = \sum_{i=1}^m \lambda_i\, \mathbb{E}_{\omega_i}[\lceil \omega_i - z_i\rceil]$$

$$= \quad \sum_{i=1}^m \lambda_i \left(\mathbb{E}_{\omega_i}\left[\lceil \omega_i - z_i\rceil^+\right] - \mathbb{E}_{\omega_i}\left[\lfloor \omega_i - z_i + 1\rfloor^-\right]\right) \qquad \text{(if } \omega \text{ continuous)}$$

- properties similar to SIR function Q . . . technical details . . .
- α-approximations with $\alpha^\star \in [0,1)^m$ optimal choice . . .

Theorem [VdV '04] If W is TU, then $Q_{\alpha^\star}$ is the convex hull of Q.
Moreover, $Q_{\alpha^\star}$ is EVF of continuous recourse problem

$$Q_{\alpha^\star}(z) = \mathbb{E}_{\xi_{\alpha^\star}}\left[\min_y \left\{qy : Wy \geq \xi_{\alpha^\star} - z,\ y \in \mathbb{R}^n_+\right\}\right]$$

with $\xi_{\alpha^\star} \sim$ cdf $\Phi_{\alpha^\star}$ discrete on $\alpha^\star + \mathbb{Z}^m$

$$\Pr\{\xi_{\alpha^\star} = \alpha^\star + l\} = \Pr\{\omega \in \textstyle\prod_{i=1}^m \left(\alpha_i^\star + l_i - 1, \alpha_i^\star + l_i\right]\}, \qquad l \in \mathbb{Z}^m$$

Example:

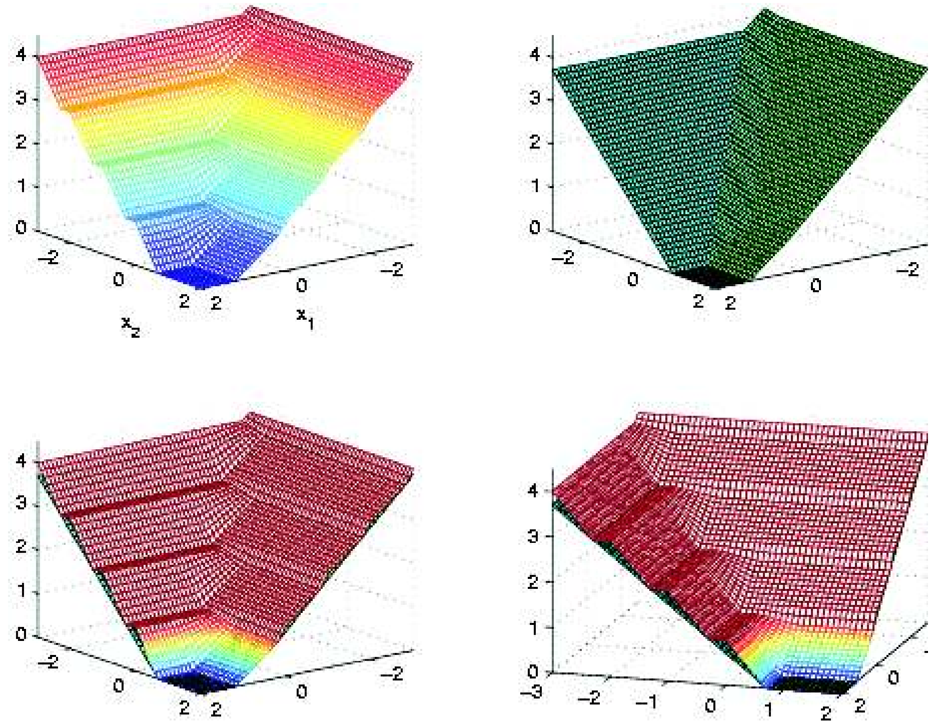
$$v(s) = \max\{\lceil s_1 \rceil, \lceil s_2 \rceil, 0\}$$

ω uniform on $(0, 0.7) \times (0, 1.2)$

$\longrightarrow \alpha^\star = (0.7,\ 0.2)$

$$\Pr\{\xi_{\alpha^\star} = (0.7, 0.2)\} = 1/6$$

$$\Pr\{\xi_{\alpha^\star} = (0.7, 1.2)\} = 5/6$$



Conclusion: to (approximately) solve TU-CIR

$$(q,\ W,\ \mathbb{Z}^m_+,\ F)$$

solve continuous CR

$$(q,\ W,\ \mathbb{R}^m_+,\ \Phi_{\alpha^\star})$$

Observe: $\Phi_{\alpha^\star}$ discrete distribution

Complete Integer Recourse (non-TU, W integral)

$$\max_{k\in\{1,\dots,K\}} \lambda^k\lceil s\rceil \leq v(s), \;\; s \in \mathbb{R}^m \longrightarrow Q_{\alpha^\star}$$ is a convex lower bound for Q

Alternative: $Q^{\mathsf{LP}}(x) := \mathbb{E}_\omega\left[\min_y\{qy : Wy \geq \omega - Tx, \; y \in \mathbb{R}^n_+\}\right]$

Theorem [VdV '04]

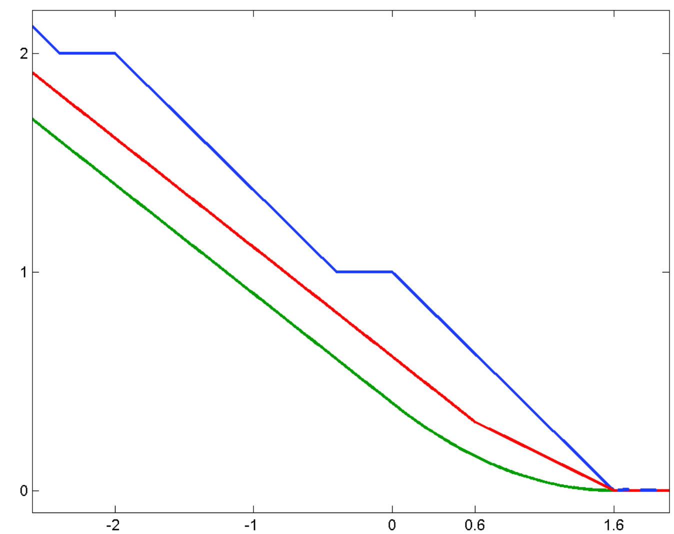
(i) $Q_{\alpha^\star} \geq Q^{\mathsf{LP}}$

(ii) Assume $q \geq 0$ $\quad (\Rightarrow Q(x) \geq 0)$

If ω continuous then

$$Q_{\alpha^\star}(x) > 0 \quad \Rightarrow \quad Q_{\alpha^\star}(x) > Q^{\mathsf{LP}}(x)$$

(Also for ω discrete $\sim$ technical conditions)



$v(s) = \min_y\{y : 2y \geq s, \; y \in \mathbb{Z}_+\}$
$T = 1 \quad \omega \sim U(0, 1.6)$

Use $Q_{\alpha^\star}$ as lower bound in other SMIP algorithms (see later)

Outline remainder

Properties two-stage complete mixed-integer recourse model

Decomposition:

- Benders
- Dual

Other approaches

Structural properties of complete mixed-integer recourse [Schultz '93 – '98]

$B \subseteq \{1,2\}$, $C \subseteq \{1,2\}$, $D \subseteq \{1,2\}$, focus on $2 \in B \cup D$

Assumptions

(a) Complete recourse

(b) Sufficiently expensive recourse

(c) $\mathbb{E}_\omega\big[|\omega|\big] < +\infty$

(a) + (b) $\Rightarrow$ v finite

(a) + (b) + (c) $\Rightarrow$ Q finite

Consider value function v

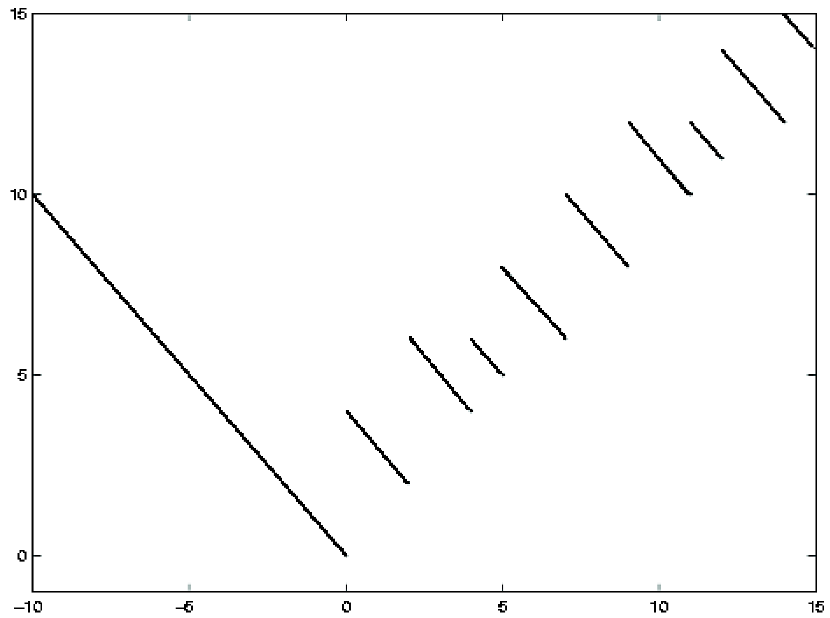
One-dimensional example

- $x \in \mathbb{R}$, $\quad T(\omega) = T = 1$
- $h(\omega) = \omega \in \mathbb{R}$ fixed

$$
\begin{aligned}
v(\omega - x) = \quad \min \quad & 2y_1 + 5y_2 + 6y_3 + y_4 \\
\text{s.t.} \quad & 2y_1 + 5y_2 + 7y_3 - y_4 = \omega - x \\
& y_1,\ y_2,\ y_3 \in \mathbb{Z}_+ \\
& y_4 \in \mathbb{R}_+
\end{aligned}
$$

Properties of v

- lower semicontinuous
- possibly discontinuous at $\omega - Tx \in \mathbb{Z}$
- size of jump varies
- piecewise linear in between jumps



Continuity:

The recourse function Q is lower semicontinuous

Define, for $x \in \mathbb{R}^n$

$$D(x) = \{\omega \in \Omega : v \text{ is discontinuous at } h(\omega) - T(\omega)x\}$$

If $\Pr\{\omega \in D(x)\} = 0$ then Q is continuous at x

$\Pr\{\omega \in D(x)\} = 0$ for all $x \in \mathbb{R}^n$ if $h(\omega) \mid T(\omega) = T$ has a pdf for a.e. T
$\longrightarrow Q$ is continuous on $\mathbb{R}^n$

Special cases:

- $h(\omega)$ and $T(\omega)$ independent, $h(\omega)$ has pdf
- $h(\omega)$ and $T(\omega)$ jointly distributed with pdf

Note: In applications, often also deterministic constraints
(e.g. flow conservation) $\longrightarrow Q$ discontinuous

(Lipschitz continuity, stability, . . .)

Large-scale mixed-integer problems

SMIP with $\Pr\left\{\big(T(\omega), h(\omega)\big) = (T^s, h^s)\right\} = p^s,\ s = 1, \ldots, S$

equivalent to large-scale deterministic MIP

$$
\begin{aligned}
\min_{x,y^s} cx + \sum_{s=1}^{S} p^s q y^s :\quad & Ax && = b \\
& T^s x + \quad W y^s && = h^s \quad \forall s \\
& x \in X, \quad y^s \in Y\ \forall s
\end{aligned}
$$

$$X = \mathbb{Z}_+^{\bar{n}} \times \mathbb{R}_+^{n-\bar{n}} \qquad Y = \mathbb{Z}_+^{\bar{p}} \times \mathbb{R}_+^{p-\bar{p}}$$

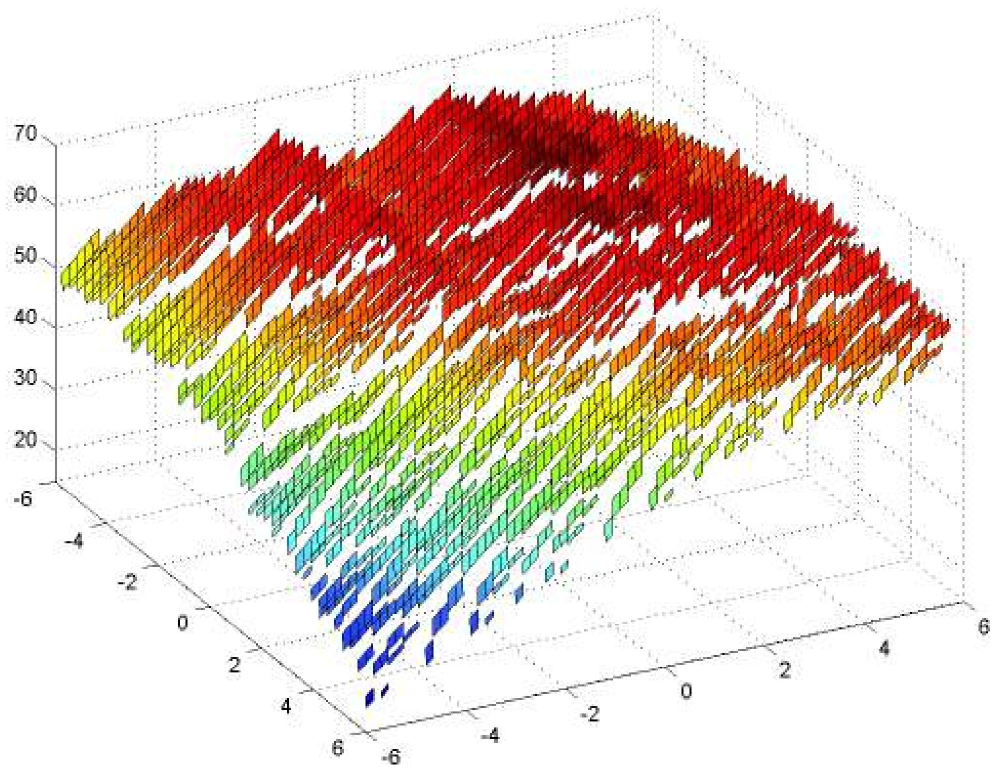
$\bar{n} + S\bar{p}$ integer variables, . . . $\longrightarrow$ For realistic values of S and $(\bar{n},)\ \bar{p}$ impossible to solve without using structure / properties

- often problem dependent
- use ideas from SLP and MIP

Stochastic multi-knapsack problem [Schultz, Stougie, VdV '98]

$$\max \left\{1.5x_1 + 4x_2 + Q(x) \; : \; x \in C = [-5, 5]^2\right\}$$

$$Q(x) := \mathbb{E}_\omega[v(\omega - Tx)],\; v(s) := \begin{array}{ll} \max & 16y_1 + 19y_2 + 23y_3 + 28y_4 \\ \text{s.t.} & 2y_1 + 3y_2 + 4y_3 + 5y_4 \leq s_1 \\ & 6y_1 + \;\; y_2 + 3y_3 + 2y_4 \leq s_2 \\ & y_i \in \{0, 1\}, i = 1, \ldots, 4 \end{array}$$



$$T = \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix}$$

$\omega \sim \mathcal{U}\{5,\, 5.5,\, \ldots,\, 14.5,\, 15\}^2$

$\longrightarrow$ MIP: 1764 Boolean, 882 constr.

Struc. Enum: optimal, CPU < 1 sec

CPLEX 5.0 – 7.1: gap 25%

$\longrightarrow$ use structure !!

CPLEX 8.1: gap 3%

Benders' decomposition and beyond

Benders is natural candidate for solving SMIP:

- a.k.a. L-shaped for solving continuous SLP [Van Slyke & Wets '69]
- Benders to solve deterministic MIP [Benders '62]

L-shaped for continuous SLP:

- complicating 1st-stage variables x: spoil separability
- for fixed x

$$Q^{\mathsf{LP}}(x) = \sum_{s=1}^{S} p^s v(\omega^s - Tx) \text{ with } v(\cdot) = \min_{y^s} \ldots$$

$\longrightarrow$ solve S deterministic LP problems in y^s (small)

Benders' decomposition for deterministic MIP:

- complicating variables $x \in \mathbb{Z}_+^p$
- fixed $x \longrightarrow$ remaining problem is LP(x)$= \min_{y \in \mathbb{R}_+^{n-p}} \ldots$

L-shaped / Benders: iterations

- solve current master problem (LP, B&B)
- generate linear optimality cuts for convex function $Q^{\mathsf{LP}}(x)$ c.q. LP(x)

Instead of $\min\{cx + Q^{\mathsf{LP}}(x) : x \in X\}$ solve iteratively

$$\min\{cx + \theta : \; x \in X, \; \theta \in \mathbb{R}$$
$$\underbrace{E_k x + \theta \geq e_k}_{\text{optimality cuts derived from } Q^{\mathsf{LP}}}, \; k = 1, \ldots, t \;\}$$

Problem: SMIP recourse function Q is non-convex

- approximate by linear cuts?
- how to derive cuts?

Wollmer [MP '80]

- 1st stage: binary 2nd stage: continuous $B = \{1\}, C = \{2\}, D = \emptyset$
- continuous second stage: Q is convex $\longrightarrow$ linear optimality cuts

Laporte & Louveaux: Integer L-shaped [ORL '93]

- 1st: binary 2nd: mixed-integer $B = \{1, 2\}, C = \{2\}, D = \{2\}$
- special class of linear optimality cuts: valid only for $x \in \{0, 1\}^n$

Carøe & Tind [MP '98]

- 1st: continuous 2nd: integer $B = \emptyset, C = \{1\}, D = \{2\}$
  (both stages mixed-integer possible)
- non-linear optimality cuts (IP duality) $\longrightarrow$ difficult master

Sen & Higle: Disjunctive Decomposition [MP '05 $\rightarrow$]

- 1st: binary 2nd: mixed-binary $B = \{1, 2\}, C = \{2\}, D = \emptyset$
- linear cuts derived from disjunctions: with $P(x, \omega)$ relaxed feasible set
  $\{y \in P : y_j \leq 0\} \cup \{y \in P : y_j \geq 1\}$

Integer L-shaped method [Laporte & Louveaux, '93]

Assumptions:

- $x \in \{0,1\}^n$ is crucial
- 2nd stage mixed-integer/binary
- $\omega \mapsto (T^s,\ h^s)$ discrete
- Q easy to compute   ($Q(x)$: solve S MIP subproblems!)

Use Branch & Cut to solve MIP master

$$\min\{cx + \theta : \ x \in X,\ \theta \in \mathbb{R}$$
$$\underbrace{E_k x + \theta \geq e_k}_{\text{optimality cuts derived from } Q},\ k = 1, \ldots, t-1\ \}$$

optimality cuts
derived from Q

adding linear optimality cuts if x^t binary

Class of optimality cuts:

Let $x^t \in X \subset \{0,1\}^n$ be the current solution

Define the index set $I^t = \{i : x_i^t = 1\} \longrightarrow$ linear cut

$$\theta \geq (Q(x^t) - L)\Big(\underbrace{\sum_{i \in I^t} x_i - \sum_{i \notin I^t} x_i}_{\leq |I^t|, \text{ with } = \text{ iff } x = x^t} - |I^t| + 1 \Big) + L$$

with L a global lower bound for Q

Observe: valid for $x \in X$, usually not for $x \in (0,1)^n$

Cuts from Q^{LP} (or Q_{α^*}) can be used

Method is finite since $X \subset \{0,1\}^n$ is finite

Numerical results: 2nd stage special structure

- capacitated vehicle routing, stochastic demands [L, L, Van Hamme '02]

Disjunctive Decomposition: sequential set convexification [Sen & Higle '05]

Assumptions

- binary 1st stage
- mixed-binary 2nd stage
- fixed recourse
- $\omega \mapsto (T^s, h^s)$ discrete

Idea

- decomposition (stagewise): allow for many scenarios
- solve only LP relaxations of subproblems (except for UB)
- strengthen subproblem LPs sequentially $\longrightarrow$ benefit in later iterations
- utilize similarities subproblems
- pass linear Benders' cuts $\sim$ strengthened subproblem LPs to master (solve by B&B)

For fixed $\bar{x}$, SMIP decomposes in S 2nd stage 0–1 MIP problems $P(\bar{x}, \omega^s)$

Solve LP relaxation $P(\bar{x}, \omega^s) \longrightarrow y^*$ with fractional value for binary y_j^*

Generate valid inequality $\pi y \geq \pi_0$, cutting of y^*

Several ways to generate valid inequalities, e.g.

- Gomory cuts
- based on Disjunctive Decomposition [Balas '79]

$$\{y \in \mathbb{R}_+^{n_2} : Wy \geq h^s - T^s x,\ y_j \leq 0\} \cup \{y \in \mathbb{R}_+^{n_2} : Wy \geq h^s - T^s x,\ y_j \geq 1\}$$

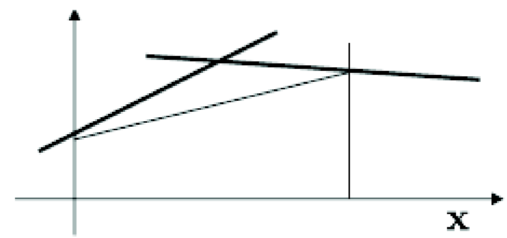
with $y_i \leq 1$ explicitly included for all binary y_i

Derived cut $\pi y \geq \pi_0$ is valid for subproblem $P(\bar{x}, \omega^s)$

Question: 'adapt' coefficients $(\pi, \pi_0) \longrightarrow$ valid inequalities for all $P(x, \omega)$?

Common Cut Coefficients (C^3) Theorem: [Higle & Sen '00]

(Conditions) There exists a function $\pi_0 : X \times \Omega \mapsto \mathbb{R}$ such that $\pi y \geq \pi_0(x, \omega)$ is valid for $P(x, \omega)$



(picture Shabbir Ahmed)

Problem: $\pi_0(x, \omega)$ is piecewise linear concave in x
$\longrightarrow$ convexify: linearize

- x binary: OK, since x extreme point of X
- harder in other cases ([Ntaimo & Sen '06]: continuous 1st stage)

In iteration k, solve auxiliary LPs to obtain

- once: π_k, appended to W_{k-1} C^{3}LP: is SLP problem itself
- for all s: α_k^s and β_k^s ($\sim \pi_0(x, \omega^s)$), appended to h_{k-1}^s and T_{k-1}^s

$\longrightarrow$ valid inequalities $\pi_k y \geq \alpha_k^s - \beta_k^s x$ for $P(x, \omega^s)$

- convergence of convex hull approximations

Using updated matrices W_k, h_k^s, T_k^s, solve LP relaxations $P(\bar{x}, \omega^s)$
$\longrightarrow$ Benders' cut to master

Related/extensions: e.g.

D^2C^3 *Algorithm* [Higle & Sen '05]

- binary 1st stage, 0-1 mixed 2nd stage
- Numerical results: e.g. server location [Ntaimo & Sen '04]

D^2-Branch-and-Cut [Sen & Sherali '06]: similar results

- binary 1st stage, general integer 2nd stage
- disjunction $\sim$ partial B&B tree for each subproblem

Cuts for deterministic equivalent problem [Carøe '98]

- continuous 1st stage, binary 2nd stage
- cuts in (x, y^s) space: translate to other scenarios

'Sequential pairing' for multi-stage SMIP [Guan, Ahmed & Nemhauser '06]

- all stages mixed-integer
- combine scenario problem cuts $\longrightarrow$ valid cut for tree problem
- numerical results: stochastic lot-sizing

Dual Decomposition in SMIP [Carøe & Schultz '99]

Assumptions:

- 1st and 2nd stage MIP $\qquad B = \{1,2\}, C = \{1,2\}, D = \{1,2\}$
- $\omega \mapsto (T^s,\ h^s)$ discrete

Applicable to multi-stage SMIP

Idea: solve S scenario problems $\sim$ realizations $(T^s,\ h^s)$

$\longrightarrow$ solutions (x^s, y^s)

$\sim$ Lagrangian relaxation of NAC: $x^1 = x^2 = \ldots = x^S$

Problem: $x^s,\ s = 1, \ldots, S$ not equal

$\longrightarrow$ use $\bar{x} = \sum p^s x^s$, but usually $\bar{x} \notin X$ (integrality)

Solution: B & B scheme

- heuristic rounding $\bar{x}$
- use Lagrangian relaxation for bounding

Details:

Deterministic equivalent MIP

$$z = \min \left\{ cx + \sum_{s=1}^{S} p^s q y^s : (x, y^s) \in C^s \ \forall s \right\}$$

where $C^s = \left\{ (x, y^s) : Ax = b, Wy^s \geq h^s - T^s x, \ x \in X, \ y^s \in Y \right\}$

Complicating variables $x \longrightarrow$ copies $x^1, \ldots, x^S$ of x

$$\min \left\{ \sum_{s=1}^{S} p^s (c x^s + q y^s) : (x^s, y^s) \in C^s \ \forall s, \ x^1 = x^2 = \ldots = x^S \right\}$$

Complicating constraints $x^1 = x^2 = \ldots = x^S \longrightarrow$ Lagrangian relaxation

Write $x^1 = x^2 = \ldots = x^S$ as $\sum_{s=1}^{S} H^s x^s = 0$

Lagrangian

$$D(\lambda) \quad = \quad \sum_{s=1}^{S} \min_{x^s, y^s} \left\{ p^s(cx^s + qy^s) + \lambda H^s x^s : (x^s, y^s) \in C^s \right\} =: \sum_{s=1}^{S} D^s(\lambda)$$

is separable $\longrightarrow$ S scenario subproblems

Lagrangian dual

$$z_{LD} := \max_{\lambda} D(\lambda), \qquad \lambda \in \mathbb{R}^{(S-1)n}$$

concave in λ, non-smooth $\longrightarrow$ subgradient methods

Weak duality: $z_{LD} \leq z$

Solve many times: S MIPs of size $(m_1 + m) \times (n + p)$

instead of once: single MIP of size $(m_1 + Sm) \times (n + Sp)$

Result: S scenario solutions (x^s, y^s), x^s not equal unless $z_{LD} = z$

Candidate first-stage solution: $\bar{x} := \sum p^s x^s \notin X$ (integrality)

$\longrightarrow$ use heuristic $[\bar{x}] :=$ round$(\bar{x}) \in X$

Branch & Bound scheme:

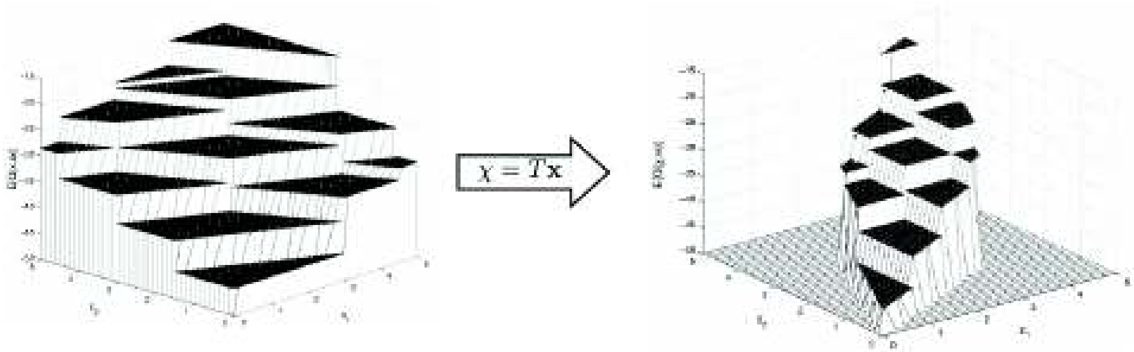
- branch on first-stage variables $\longrightarrow$ equality of $x^s,\ s = 1, \ldots, S$ (partition X if x_i continuous)
- use Lagrangian relaxation for bounding

Numerical results: realistic unit commitment problem, . . .

With modifications also for several mean-risk models
[Schultz & co-authors '03 $\rightarrow$]

Decomposition based B&B [Ahmed, Tawarmalani, Sahinidis '04]

T fixed, $q(\omega), W(\omega), h(\omega)$ discrete, $B = \{1,2\}, C = \{1\}, D = \{1,2\}$



(picture: Shabbir Ahmed)

SMIP problem in space of tender variables:

$$\min\{f(\chi) + \sum_{s=1}^{S} p^s v(\chi - h^s) : \chi \in \mathcal{X}\}$$

with $f(\chi) := \min\{cx : x \in X,\ Tx = \chi\}$ and $\mathcal{X} := \{\chi : Tx = \chi,\ x \in X\}$

Discontinuities orthogonal to tender axes: allows to solve by B&B

- Branching: partition along discontinuities
- LB: structural properties of v (building on [Schultz, Stougie, VdV '98])
- UB: function evaluation

Branch-and-Fix Coordination [Alonso-Ayuso, Escudero, Ortuño '03]

Applies to multi-stage mixed-binary SMIP:
$B = \{1, 2, \ldots, H\}, C = \{1, 2, \ldots, H\}, D = \emptyset$

Key ideas:

- Lagrange relaxation NAC $\longrightarrow$ solve scenario MIP problems by B&B
- use NAC to coordinate branching

Implementation: very technical (bookkeeping)

Numerical results: realistic size, e.g. supply chain planning

Concluding remarks

SMIP has many applications

Lots of challenges!

Computational results for real-life problems

Solution methods for multi-stage, risk models, . . .

Settle for sub-optimal results

- excused by complexity: SMIP = SLP $\times$ MIP
- improvement over simpler models

Use of problem-specific structure $\longrightarrow$ algorithms/heuristics
($\sim$ deterministic MIP)

- plenary David Morton (Mon 2:00–3:00): network interdiction