Chance Constrained Problems

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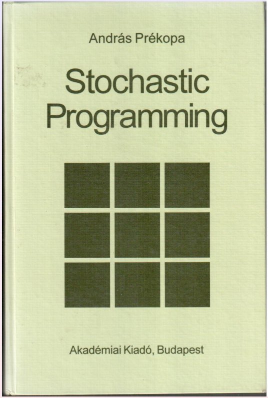
August 15, 2009

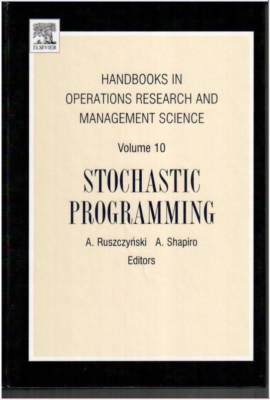
Suggested Reading



András Prékopa

Stochastic Programming

Akadémiai Kiadó, Budapest



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HANDBOOKS IN
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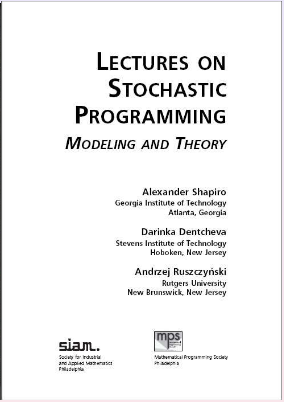
Volume 10

STOCHASTIC
PROGRAMMING

A. Ruszczyński A. Shapiro
Editors



LECTURES ON
STOCHASTIC
PROGRAMMING

MODELING AND THEORY

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Contents

Models

Optimization problems with random constraints

$$min\{f(x)|g(x,\xi) \geq 0\} \tag{1}$$

x = decision vector

ξ = random vector

f = objective

g = constraint mapping

Example (Random parameters)

- Random returns in portfolio optimization
- Random meteorological data (precipitation, temperature) in power production
- Random demands in power, gas, water networks
- Position of random obstacles in robotics
- Random demographical data in pension fund management

Typical Situation: Taking a decision before observing the random vector

$\Rightarrow$ (1) not solvable $\Rightarrow$ find deterministic reformulations!

Deterministic Reformulations

Optimization problem with random constraints: $min\{f(x)|g(x,\xi)\geq 0\}$

- EXPECTATION CONSTRAINTS: $min\{f(x)|g(x,\mathbb{E}\xi)\geq 0\}$

  Pro: easy to solve, solutions at low costs Con: solutions not robust

- WORST-CASE CONSTRAINTS: $min\{f(x)|g(x,\xi)\geq 0 \quad \forall\xi\}$

  Pro: absolutely robust solutions Con: solutions extremely expensive or do not even exist

- CHANCE CONSTRAINTS:

$$min\{f(x)|\underbrace{\mathbb{P}(g(x,\xi)\geq 0)}_{\varphi(x)}\geq p\} \quad p\in[0,1]$$

Pro: robust solutions, not too expensive Con: often difficult to solve

Numerous applications in engineering

Challenge: φ not explicit. Structure? Numerics? Stability?

Linear Chance Constraints

General chance constraint: $\mathbb{P}(g(x,\xi) \geq 0) \geq p$.

If g is linear in ξ, the chance constraint is called linear.

Two types:

Type I (separable model): $\boxed{\mathbb{P}(h(x) \geq A\xi) \geq p}$.

Example: Capacity optimization in stochastic networks ▸ Insertion

Type II (bilinear model): $\boxed{\mathbb{P}(\Xi \cdot x \geq b) \geq p}$

Example: Mixture problems ▸ Insertion

Random right-hand side

Special case of the separable model: random right-hand side

$$\mathbb{P}(h(x) \geq \xi) \geq p.$$

Using the distribution function $F_\xi(z) := \mathbb{P}(\xi \leq z)$, the constraint can be rewritten as

$$F_\xi(h(x)) \geq p.$$

$\Longrightarrow$ Exploit knowledge about analytical properties and numerical computation of multivariate distribution functions!

If ξ has a density f_ξ, then

$$F_\xi(z) = \int_{-\infty}^{z_1} \cdots \int_{-\infty}^{z_s} f_\xi(x) dx_s \cdots dx_1$$

Joint vs. individual chance constraints

When dealing with a random inequality system

$$g_i(x,\xi) \geq 0 \quad (i = 1,\dots,s),$$

one has two options for generating chance constraints:

- Joint chance constraint: $\mathbb{P}(g_i(x,\xi) \geq 0 \quad (i = 1,\dots,s)) \geq p$
- Individual chance constraints: $\mathbb{P}(g_i(x,\xi) \geq 0) \geq p \quad (i = 1,\dots,s)$

Random right-hand side: $g_i(x,\xi) = h_i(x) - \xi_i$.

Model with indiv. chance constraints maintains deterministic simplicity:

$$\mathbb{P}(h_i(x) \geq \xi_i) \geq p \quad (i = 1,\dots,s) \Longleftrightarrow h_i(x) \geq \underbrace{q_i(p)}_{p-\text{quantile of } \xi_i} \quad (i = 1,\dots,s)$$

$\Longrightarrow$ (sometimes) a strong numerical simplification but often the wrong model

Structural properties

Structural properties of chance constraints

Information about structural properties of the probability function

$$\varphi(x) := \mathbb{P}(g(x,\xi) \geq 0)$$

and of the induced set of feasible decisions

$$M := \{x \in \mathbb{R}^n \mid \varphi(x) \geq p\}$$

is essential for the design of algorithms.

Proposition (Upper semicontinuity, closedness)

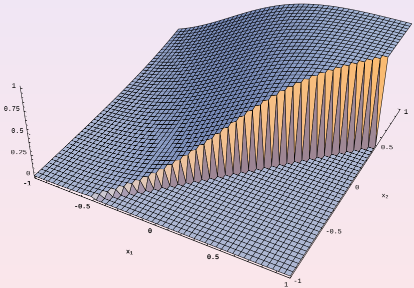
If the g_i are usc., then so is φ. As a consequence, M is closed.

No properties beyond usc are evident.

A counter example for continuity

$$\varphi(x) := \mathbb{P}(Mx + L\xi \geq b); \quad \xi \sim \text{one-dimensional standard normal}$$

$$(M|L) = \left(\begin{array}{cc|c} 2 & 1 & -1 \\ -1 & 1 & 0 \end{array} \right) \quad b = \left(\begin{array}{c} 0 \\ -0.5 \end{array} \right)$$



$g(x, \xi)$ affine linear

distribution of ξ smooth

φ not continuous!

Continuity of Chance Constraints

As before, let

$$\varphi(x) := \mathbb{P}(g(x,\xi) \geq 0)$$

Proposition (Raik 1971)

If the g_i are continuous and, additionally,

$$\mathbb{P}\left(g_i\left(x,\xi\right) = 0\right) = 0 \quad \forall x \in \mathbb{R}^n \quad \forall i \in \{1,\ldots,s\},$$

then φ is continuous too.

Analytical Properties of distribution functions

Let ξ be an s-dimensional random vector with distribution function F_ξ

Proposition

If ξ has a density f_ξ, i.e., $F_\xi(z) = \int_{-\infty}^{z} f_\xi(x)dx$, then F_ξ is continuous.

Theorem (Wang 1985, Römisch/Schultz 1993)

If ξ has a density f_ξ, then F_ξ is Lipschitz continuous if and only if all marginal densities f_{ξ_i} are essentially bounded.

▸ Insertion

Theorem (R.H./Römisch 2010)

If ξ has a density f_ξ such that $f_\xi^{-1/s}$ is convex, then F_ξ is Lipschitz continuous.

Assumption satisfied by most prominent multivariate distributions:
Gaussian, Dirichlet, t, Wishart, Gamma, lognormal, uniform

Continuous differentiability of distribution functions

Conjecture

If ξ has a continuous density f_ξ such that all marginal densities f_{ξ_i} are continuous too, then F_ξ is continuously differentiable.

Conjecture wrong !

▸ Insertion

Proposition

Let $z \in \mathbb{R}^s$ be given. If all one-dimensional functions

$$\varphi^{(i)}(t) := \int\limits_{-\infty}^{z_1} \cdots \int\limits_{-\infty}^{z_{i-1}} \int\limits_{-\infty}^{z_{i+1}} \cdots \int\limits_{-\infty}^{z_s} f_\xi\left(u_1, \ldots, u_{i-1}, t, u_{i+1}, \ldots, u_s\right) du_1 \cdots du_{i-1}, du_{i+1} \cdots du_s$$

are continuous, then the partial derivatives of F_ξ exist and it holds that

$$\frac{\partial F_\xi}{\partial z_i}(z) = \varphi^{(i)}(z_i).$$

Example: Derivative of Gaussian distribution function

Example

If $\xi \sim \mathcal{N}(\mu, \Sigma)$ with Σ positive definite, then

$$\frac{\partial F_{\xi}}{\partial z_i}(z) = f_{\xi_i}(z_i) \cdot F_{\tilde{\xi}(z_i)}(z_1, \ldots, z_{i-1}, z_{i+1} \ldots, z_s) \quad (i = 1, \ldots, s)\,.$$

Here, $f_{\xi_i} =$ density of ξ_i (1-dimensional Gaussian) and $\tilde{\xi}(z_i) \sim \mathcal{N}(\tilde{\mu}, \tilde{\Sigma})$ with

$$\begin{aligned} \tilde{\mu} &= D_i \left(\mu + \Sigma_{ii}^{-1} (z_i - \mu_i) \Sigma_i \right) \\ \tilde{\Sigma} &= D_i \left(\Sigma - \Sigma_{ii}^{-1} \Sigma_i \Sigma_i^T \right) D_i^T \\ D_i &= \text{identity matrix with row } i \text{ removed} \\ \Sigma_i &= \text{row } i \text{ of } \Sigma \end{aligned}$$

Computation of the gradient is analytically reduced to the computation of functional values!

For second partial derivatives proceed by induction.

Convexity

Convexity of the feasible set in the separable model

Linear chance constraint with random right-hand side: $\boxed{\mathbb{P}(h(x) \geq \xi) \geq p}$

Equivalent description with distribution function: $\boxed{F_\xi(h(x)) \geq p}$ $(*)$

When is $F_\xi \circ h$ concave? $\Longrightarrow$ algorithms of convex optimization!

sufficient:

- components h_j concave $\Longrightarrow$ e.g., h is a linear mapping
- F_ξ increasing $\Longrightarrow$ automatic (distribution function)
- F_ξ concave $\Longrightarrow$ never! (distribution function)

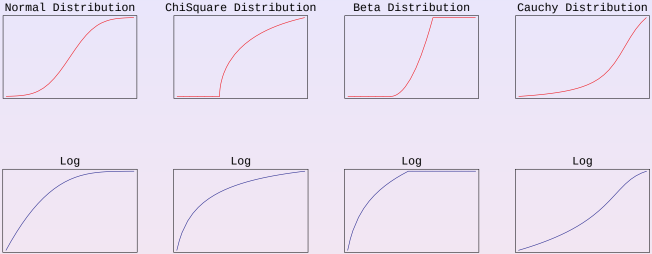
Is there a strictly increasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}$ such that $\varphi \circ F_\xi$ is concave?

If so, then $(*) \Longleftrightarrow \varphi(F_\xi(h(x))) \geq \varphi(p)$.

$\underbrace{\varphi \circ F_\xi}_{\substack{\text{increasing}\\ \text{concave}}} \circ h$ is concave if the components h_j are concave.

Potential candidates: $\varphi = \log$, $\varphi = -(\cdot)^{-n}$

Log-concavity of distribution functions



Most (not all) prominent distributions are log-concave.

Often easy to verify in one dimension

No chance to do so explicitly in several dimensions (try simple case of uniform distribution on a ball).

Prékopa's Theorem (reduced version)

Theorem (Prékopa 1973)

Log-concavity of the density implies log-concavity of the distribution function.

Example (normal distribution)

$$\begin{aligned} f_\xi(x) &= K \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right) \\ \log f_\xi(x) &= \log K - \frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu) \quad \Longrightarrow \log F_\xi \text{ concave} \end{aligned}$$

Example (other examples for log-concave distributions)

Gaussian, Dirichlet, Student, lognormal, Gamma, uniform, Wishart

Corollary (Convexity in the separable model)

Consider the feasible set $M := \{x \in \mathbb{R}^n \mid \mathbb{P}(A\xi \le h(x)) \ge p\}$*. Let* ξ *have a density* f_ξ *such that* $\log f_\xi$ *is concave and let the* h_i *be concave (e.g., h linear). Then, M is convex for any* $p \in [0,1]$*.*

Convexity of the feasible set in the bilinear model

Consider the feasible set $M := \{x \in \mathbb{R}^n \mid \mathbb{P}(\Xi x \leq a) \geq p\}$

Theorem (Van de Panne/Popp, Kataoka 1963, Kan 2002, Lagoa/Sznaier 2005)

Let Ξ have one row only which has an elliptically symmetric or log-concave symmetric distribution (e.g., Gaussian). Then, M is convex for $p \geq 0.5$.

Theorem (R.H./Strugarek 2008)

Let the rows ξ_i of Ξ be Gaussian according to $\xi_i \sim \mathcal{N}(\mu_i, \Sigma_i)$. If the ξ_i are pairwise independent, then M is convex for $p > \Phi(\max\{\sqrt{3}, \tau\})$, where

$$\begin{array}{rcl} \Phi & = & \textit{1-dimensional standard normal distribution function} \\ \tau & := & \max\limits_i \lambda_{max}^{(i)} [\lambda_{min}^{(i)}]^{-3/2} \|\mu_i\| \\ \lambda_{max}^{(i)}, \lambda_{min}^{(i)} & := & \textit{largest and smallest eigenvalue of } \Sigma_i. \end{array}$$

Moreover, M is compact for $p > \min_i \Phi(\|\mu_i\|_{\Sigma_i^{-1}})$.

Numerical Aspects for Problems with Gaussian Data

Random right-hand side with Gaussian data

Model with random right-hand side:

$$\mathbb{P}(h(x) \geq \xi) \geq p \quad \Longrightarrow \quad F_\xi(h(x)) \geq p.$$

If $\xi \sim \mathcal{N}(\mu, \Sigma)$ with Σ positive definite (regular Gaussian), then

$$\frac{\partial F_\xi}{\partial z_i}(z) = f_{\xi_i}(z_i) \cdot F_{\tilde{\xi}(z_i)}(z_1, \ldots, z_{i-1}, z_{i+1} \ldots, z_s) \quad (i = 1, \ldots, s)\,.$$

Use efficient method to compute F_ξ (and thus ∇F_ξ).

E.g., code by A. Genz. Computes Gaussian probabilities of rectangles:

$$\mathbb{P}(\xi \in [a, b]) \qquad (F_\xi(z) = \mathbb{P}(\xi \in (-\infty, z])$$

Allows to consider problems with up to a few hundred random variables.

Can we benefit from this tool in order to cope with more complicated models?

$$\mathbb{P}(h(x) \geq A\xi) \geq p, \qquad \mathbb{P}(\Xi \cdot x \geq b) \geq p$$

Derivatives for Gaussian probabilities of rectangles I

Let $\xi \sim \mathcal{N}(\mu, \Sigma)$ with Σ positive definite.

Consider a two-sided probabilistic constraint:

$\mathbb{P}(\xi \in [a(x), b(x)]) \geq p$.   This may be written as

$$\alpha_\xi(a(x), b(x)) \geq p, \quad \text{where} \quad \alpha_\xi(a, b) := \mathbb{P}(\xi \in [a, b])$$

How to compute partial derivatives $(\partial \alpha_\xi / \partial a, b)$?

First naive approach:

Reduction to distribution functions, then use known gradient formula.

$$\alpha_\xi(a, b) = \sum_{i_1, \dots, i_s \in \{0,1\}} (-1)^{\left[s + \sum_{j=1}^{s} i_j\right]} F_\xi(y_{i_1}, \dots, y_{i_s}), \quad y_{i_j} := \left\{ \begin{array}{ll} a_j & \text{if } i_j = 0 \\ b_j & \text{if } i_j = 1 \end{array} \right.$$

In dimension s, there are 2^s terms in the sum. Not practicable!

Second naive approach:

$$\alpha_\xi(a, b) = \mathbb{P}\left(\begin{pmatrix} \xi \\ -\xi \end{pmatrix} \leq \begin{pmatrix} b \\ -a \end{pmatrix}\right), \quad \begin{pmatrix} \xi \\ -\xi \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu \\ -\mu \end{pmatrix}, \begin{pmatrix} \Sigma & -\Sigma \\ -\Sigma & \Sigma \end{pmatrix}\right)$$

$\Longrightarrow$ Singular normal distribution, gradient formula not available.

Derivatives for Gaussian probabilities of rectangles II

Proposition (Ackooij/R.H./Möller/Zorgati 2010)

Let $\xi \sim \mathcal{N}(\mu, \Sigma)$ *with* Σ *positive definite and* f_ξ *the corresponding density.*

Then,

$$\frac{\partial \alpha_\xi}{\partial b_i}(a,b) = f_{\xi_i}(b_i)\, \alpha_{\tilde{\xi}}(b_i)(\tilde{a}, \tilde{b}); \qquad \frac{\partial \alpha_\xi}{\partial a_i}(a,b) = -f_{\xi_i}(a_i)\, \alpha_{\tilde{\xi}}(a_i)(\tilde{a}, \tilde{b})$$

with the tilda-quantities defined as in the gradient formula for Gaussian distribution functions.

$\Longrightarrow$ Use Genz' code to calculate α_ξ and $\nabla_{a,b}\alpha_\xi$ at a time.

Allows to consider problems in similar dimension as for pure random right-hand side.

Derivatives for the separated model with Gaussian data

Let $\xi \sim \mathcal{N}(\mu, \Sigma)$ with Σ positive definite and consider the probability function

$$\beta_{\xi,A}(x) := \mathbb{P}(A\xi \leq x)$$

If the rows of A are linearly independent, then put $\eta := A\xi \sim \mathcal{N}(A\mu, A\Sigma A^T)$

$$\Longrightarrow \quad \beta_{\xi,A}(x) = F_\eta(x) \quad \text{regular Gaussian distribution function}$$

Otherwise (e.g, capacity optimization in stochastic networks):

F_η is a singular Gaussian distribution function (gradient formula not available).

Theorem (R.H./Möller 2010, see talk by A. Möller, Thursday, 3.20 p.m., R. 1020)

$$\frac{\partial \beta_{\xi,A}}{\partial x_i}(x) = f_{A_i\xi}(x_i)\beta_{\tilde{\xi},\tilde{A}}(\tilde{x}),$$

where $\tilde{\xi} \sim \mathcal{N}(0, I_{s-1})$ and $\tilde{A}, \tilde{x}$ can be calculated explicitly from A and x.

Use, e.g., Deák's code for calculating normal probabilities of convex sets.

Derivatives for the bilinear model with Gaussian data

Consider the probability function

$$\gamma(x) := \mathbb{P}(\Xi x \leq a)$$

with normally distributed (m, s) coefficient matrix. Let ξ_i be the ith row of Ξ.

Proposition (R.H./Möller 2010)

$$\begin{aligned} \gamma(x) &= F_\eta(a) \\ \nabla\gamma(x) &= \sum_{i=1}^{m} \frac{\partial F_\eta}{\partial z_i}(\beta(x))\nabla\beta_i(x) + \sum_{i,j=1}^{m} \frac{\partial^2 F_\eta}{\partial z_i \partial z_j}(\beta(x))\nabla R_{ij}(x), \end{aligned}$$

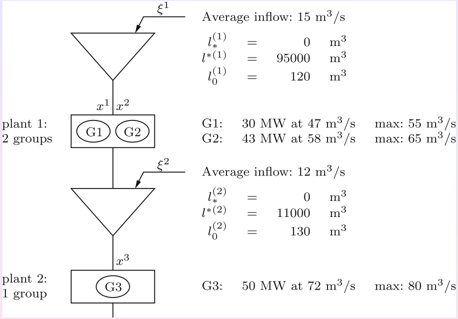
$$\begin{aligned} \eta &\sim \mathcal{N}(0, R(x)) \\ \mu(x) &= \left(\mathbb{E}\xi_i^T x\right)_{i=1}^{m} \\ \Sigma(x) &= \left(x^T \mathrm{Cov}\,(\xi_i, \xi_j) x\right)_{i,j=1}^{m} \\ D(x) &= \mathrm{diag}\,\left(\Sigma_{ii}^{-1/2}(x)\right)_{i=1}^{m} \\ R(x) &= D(x)\Sigma(x)D(x) \\ \beta(x) &= D(x)(a - \mu(x)) \end{aligned}$$

An Example

Example: Hydro Power Management (Model)

Data: Electricité de France R&D

Ackooij/R.H./Möller/Zorgati 2010



Average inflow: 15 m^3/s

$l_*^{(1)} = 0$ m^3

$l^{*(1)} = 95000$ m^3

$l_0^{(1)} = 120$ m^3

G1: 30 MW at 47 m^3/s max: 55 m^3/s

G2: 43 MW at 58 m^3/s max: 65 m^3/s

Average inflow: 12 m^3/s

$l_*^{(2)} = 0$ m^3

$l^{*(2)} = 11000$ m^3

$l_0^{(2)} = 130$ m^3

G3: 50 MW at 72 m^3/s max: 80 m^3/s

Detailed Model

Time-dependent filling levels in reservoirs:

$$l_t^{(1)} = l_0^{(1)} + \sum_{\tau=1}^{t} \xi_\tau^{(1)} - \sum_{\tau=1}^{t} x_\tau^{(1)} - \sum_{\tau=1}^{t} x_\tau^{(2)}$$

$$l_t^{(2)} = l_0^{(2)} + \sum_{\tau=1}^{t} \xi_\tau^{(2)} + \sum_{\tau=1}^{t} x_\tau^{(1)} + \sum_{\tau=1}^{t} x_\tau^{(2)} - \sum_{\tau=1}^{t} x_\tau^{(3)}$$

Objective function:

$$\sum_{j=1}^{3}\sum_{t=1}^{T} \underbrace{\lambda^{(j)}\pi_t x_t^{(j)}}_{\text{profit by sale}} + \mathbb{E} \underbrace{\omega_1 l_T^{(1)} + \omega_2 l_T^{(2)}}_{\text{evaluation of final water level}}$$

Abstract optimization problem with probabilistic constraints:

Joint constraints $\min\{c^T x \mid \mathbb{P}(Ax + a \le L\xi \le Bx + b) \ge p,\ \ x \in [0, x^{\max}]\}$

Individual constraints
$$\min\left\{ c^T x \;\middle|\; \begin{array}{rcl} \mathbb{P}(A_i x + a_i \le L_i\xi) & \ge & p, \\ \mathbb{P}(L_i\xi \le B_i x + b_i) & \ge & p \quad (i = 1, \ldots, T) \\ x \in [0, x^{\max}] & & \end{array} \right\}$$

Example: Hydro Power Management (Problem data and solution methods)

Problem data:

Time horizon: $T = 32$ (8 hours in 15 min. steps)

Probability level: $p = 0.9$

Gaussian inflow process: $\xi := (\xi_1, \xi_2) \sim \mathcal{N}(\mu, \Sigma)$, Σ positive definite

$\Longrightarrow$ feasible set convex and $\eta := L\xi \sim \mathcal{N}(L\mu, L\Sigma L^T)$ regular.

$\Longrightarrow$ Two-sided probabilistic constraint: $\mathbb{P}(a(x) \leq \eta \leq b(x)) \geq p$

Solution Methods:

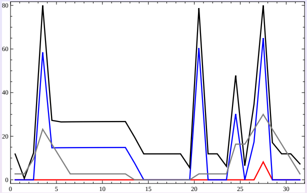
Joint probabilistic constraints: cutting plane method using calculus for values and gradients of Gaussian probabilties of rectangles

Individual probabilistic constraints: Linear programming

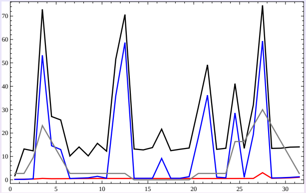
Example: Hydro Power Management (Results 1)

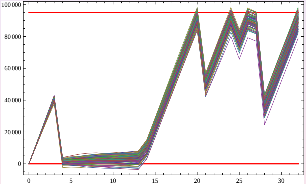
Individual probabilistic constraints

Joint probabilistic constraints

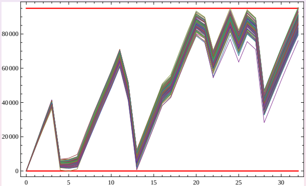


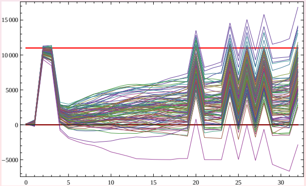
Optimal release for the three turbines (coloured) and price signal (gray)



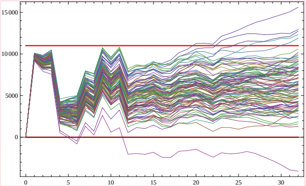


100 simulated filling levels in upper reservoir





100 simulated filling levels in lower reservoir



Example: Hydro Power Management (Results 2)

Model	#violating scenarios	Optimal value
joint constraints	10	4803072
individual constraints	68	4954845

Individual probabilistic constraints satisfy level constraints with $p = 0.9$ at each time $t = 1, \ldots, T$ but:

probability of satisfying level constraints through the whole interval is only $p \approx 0.32$.

$\Longrightarrow$ Though easy to solve, model with individual constraints is not appropriate in general.

For model with dynamic chance constraints see talk by R.H., Tuesday, 4.35 p.m., R. 1028

Stability

Optimization problem: $\min\{f(x) \mid x \in C,\ \mathbb{P}(\xi \leq Ax) \geq p\}$

Distribution of ξ rarely known $\Longrightarrow$ Approximation by some $\eta \Longrightarrow$ Stability?

Solution set mapping: $\Psi(\eta) := \mathrm{argmin}\,\{f(x) \mid x \in C,\ \mathbb{P}(\eta \leq Ax) \geq p\}$

Theorem (R.H./W.Römisch 2004)

- *f convex, C convex, closed, ξ has log-concave density*
- *$\Psi(\xi)$ nonempty and bounded*
- *$\exists x \in C: \quad \mathbb{P}(\xi \leq Ax) > p$ (Slater point)*

Then, Ψ is upper semicontinuous at ξ:

$$\Psi(\eta) \subseteq \Psi(\xi) + \varepsilon\mathbb{B} \quad \text{for} \quad \sup_{z \in \mathbb{R}^s} |F_\xi(z) - F_\eta(z)| < \delta$$

If in addition

- *f convex-quadratic, C polyhedron,*
- *ξ has strongly log-concave distribution function,*

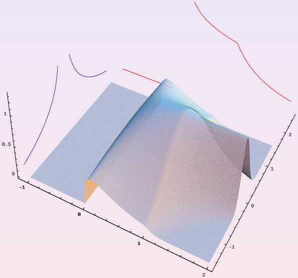
then Ψ is locally Hausdorff-Hölder continuous at ξ:

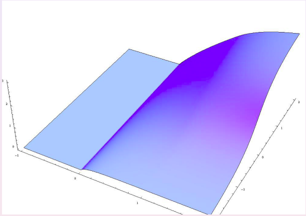
$$d_{\mathrm{Haus}}(\Psi(\eta), \Psi(\xi)) \leq L\sqrt{\sup_{z \in \mathbb{R}^s} |F_\xi(z) - F_\eta(z)|} \quad \text{(locally around } \xi\text{)}$$

End

A counter example for Lipschitz continuity (A. Wakolbinger)

Let ξ have the density $$f_\xi(x,y) := \begin{cases} 0 & x < 0 \\ cx^{1/4}e^{-xy^2} & x \in [0,1] \\ ce^{-x^4y^2} & x > 1 \end{cases}$$





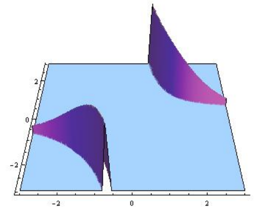
Then, f_ξ is bounded (even continuous) but f_{ξ_1}, f_{ξ_2} are not.

$\Longrightarrow$ F_ξ fails to be Lipschitz continuous.

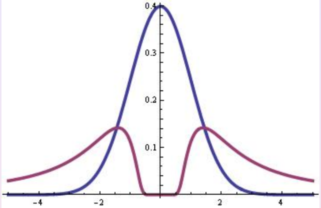
◂ back

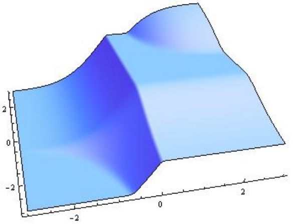
A counter example for differentiability (R.H. 2010)

Let ξ have the density $f_\xi(x,y) := \begin{cases} \max\{0, \frac{1}{\sqrt[4]{\pi}} e^{-x^2/4} - |y - 2/x|\} & \text{if } x \neq 0 \\ 0 & \text{else} \end{cases}$



density and both marginal densities are continuous and bounded



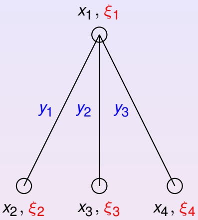


Distribution function not differentiable

back

Capacity optimization in a stochastic network

Stochastic Network (see, e.g., Prékopa 1996)



ξ_i — stochastic demand

x_i — generator capacities

y_i — transmission capacities

Inequality system describing satisfaction of demand:

$$\begin{aligned}
\xi_1+\xi_2+\xi_3+\xi_4 &\le x_1+x_2+x_3+x_4\\
\xi_1 &\le x_1+y_2+y_3+y_4\\
\xi_2 &\le x_2+y_2\\
\xi_3 &\le x_3+y_3\\
\xi_4 &\le x_4+y_4\\
\xi_1+\xi_2 &\le x_1+x_2+y_3+y_4\\
\xi_1+\xi_3 &\le x_1+x_3+y_2+y_4\\
\xi_1+\xi_4 &\le x_1+x_4+y_2+y_3\\
\xi_1+\xi_2+\xi_3 &\le x_1+x_2+x_3+y_4\\
\xi_1+\xi_2+\xi_4 &\le x_1+x_2+x_4+y_3\\
\xi_1+\xi_3+\xi_4 &\le x_1+x_3+x_4+y_2
\end{aligned}$$

Optimisation problem:

$$min\{\underset{\substack{\uparrow\\ \text{costs}}}{f(x,y)} \mid \mathbb{P}(L\xi \le \underset{\substack{\uparrow\\ \text{satisfaction of demand at}\\ \text{probability p}}}{Ax+By}) \ge p\}$$

◂ back

Mixture problems

Example

- Blending problems: Blend raw materials (scrap, tungsten ores, coffees, nutrients), such that certain random contaminations (e.g., heavy metals) or -agents (e.g., flavoring substances) meet some lower concentration limit in the arising mixture at high probability.

$$\mathbb{P}(\Xi \cdot x \geq b) \geq p$$

$$\Xi = \begin{array}{cc} \xrightarrow{\text{raw materials}} & \\ \begin{pmatrix} \xi_{1,1} & \cdots & \xi_{1,n} \\ \vdots & \ddots & \vdots \\ \xi_{m,1} & \cdots & \xi_{m,n} \end{pmatrix} & \Big\downarrow \text{concentrations} \end{array} \qquad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \Big\downarrow \text{Mixture} \qquad b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \Big\downarrow \text{lower limits}$$

back

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